Lecture 1: Addition of Angular Momentum

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1. **Review: Angular Momentum Algebra**
   - Definitions: $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$; basic commutators $[x_i, p_j] = i\hbar\,\delta_{ij}$.
   - $[L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k$; consequences for a general $\boldsymbol{J}$.
   - Eigenvalue problems: $J^2\,|j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle,\; J_z\,|j,m\rangle = \hbar m\,|j,m\rangle$.
   - Ladder operators: $J_\pm = J_x \pm iJ_y$ and their action; identities involving $J_\pm$.
   - *Exercise:* Construct J_x, J_y, J_z matrices for $j = 9/2$ and verify commutators.
2. **Coupling of Angular Momentum**
   - Total angular momentum: $\boldsymbol{J} = \boldsymbol{J}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \boldsymbol{J}_2$; components $J_i = J_{1,i} \otimes \mathbb{I} + \mathbb{I} \otimes J_{2,i}$.
   - Uncoupled CSCO: $J_1^2, J_2^2, J_{1z}, J_{2z}$ with basis $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$.
   - Coupled CSCO: J_1^2, J_2^2, J^2, J_z with basis $|j_1, j_2; j, m\rangle$; allowed j: $|j_1 - j_2| \le j \le j_1 + j_2$.
   - Clebsch–Gordan coefficients $C^{jm}_{j_1 m_1,\, j_2 m_2}$; selection rule $m = m_1 + m_2$.
   - Relation to Wigner $3j$ symbols; symmetry/phase rules and key orthogonality sum rules.
   - *Exercise:* Build matrix representations of $\mathbf{I}\cdot\mathbf{J}$ for $I = 9/2,\ J = 1$ in both bases.
3. **Spherical Basis**
   - Basis vectors: $\boldsymbol{\epsilon}^{(1)}_{\pm 1} = \mp(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}$, $\boldsymbol{\epsilon}^{(1)}_0 = \hat{\mathbf{z}}$.
   - Component transforms: $A^{(1)}_{\pm 1} = \mp(A_x \pm iA_y)/\sqrt{2}$, $A^{(1)}_0 = A_z$.
   - Position operator in spherical basis; links to $Y_1^m(\theta, \phi)$.
   - Cross products in spherical basis; handy identity via a single $3j$ symbol.
4. **Symmetry Transformations of AM States and Operators**
   - Parity $\mathcal{P}$: unitary/Hermitian; eigenvalues ± 1; action on Y_ℓ^m and $|\ell, m_\ell\rangle$; momentum inversion.
   - Time reversal $\mathcal{T}$: antiunitary; flips spin and $i \to -i$ (overview).
   - Rotations: Euler decomposition $R(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$.
   - Wigner D and d matrices; rotation of states $|j, m\rangle$ and spherical tensors $T_q^{(k)}$.
   - Building $D^{(j)}$ from lower-j via CG; unitarity and phase properties.
5. **Summary and Practice**
   - AM as the generator of rotations; coupling rules; $3j$ symmetry and sum rules; spherical basis toolbox.
   - *Exercise (atom–light coupling):* Using CG symmetry, infer relative signs of Rabi frequencies for $F = 1 \to F' = 1, 2$ transitions across all $m_F \to m'_F$.